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Anbar Meidl, Nurdagül; Odzak, Almasa; Patel, Vandita; Quoos, Luciane; Somoza, Anna; Topuzoglu, Alev

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On the difference between permutation polynomials over finite fields

Nurdagül Anbar[1], Almasa Odžak[2], Vandita Patel[3], Luciane Quoos[4], Anna Somoza[5,6], Alev Topuzoğlu[7]

[1] *Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, 4040-Linz, Austria*
Email: nurdagulanbar2@gmail.com
[2] *University of Sarajevo, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina*
Email: almasa.odzak@gmail.com
[3] *University of Warwick, Coventry CV4 7AL, UK*
Email: vandita.patel@warwick.ac.uk
[4] *Universidade Federal do Rio de Janeiro, Cidade Universitária, Rio de Janeiro, RJ 21941-909, Brazil*
Email: luciane@im.ufrj.br
[5] *Universitat Politècnica de Catalunya, Calle Jordi Girona, 1-3, 08034 Barcelona, Spain*
[6] *Leiden University , Snellius building, Niels Bohrweg 1 2300 RA Leiden, Netherlands*
Email: anna.somoza@upc.edu
[7] *SabancıUniversity, MDBF, Orhanlı, Tuzla, 34956 İstanbul, Turkey*
Email: alev@sabanciuniv.edu

Abstract

The well-known Chowla and Zassenhaus conjecture, proven by Cohen in 1990, states that if $p > (d^2 - 3d + 4)^2$, then there is no complete mapping polynomial f in $\mathbb{F}_p[x]$ of degree $d \geq 2$. For arbitrary finite fields $\mathbb{F}_q$, a similar non-existence result is obtained recently by Işık, Topuzoğlu and Winterhof in terms of the Carlitz rank of f.

Cohen, Mullen and Shiue generalized the Chowla-Zassenhaus-Cohen Theorem significantly in 1995, by considering differences of permutation polynomials. More precisely, they showed that if f and $f+g$ are both permutation polynomials of degree $d \geq 2$ over $\mathbb{F}_p$, with $p > (d^2-3d+4)^2$, then the degree k of g satisfies $k \geq 3d/5$, unless g is constant. In this article, assuming f and $f+g$ are permutation polynomials in $\mathbb{F}_q[x]$, we give lower bounds for k in terms of the Carlitz rank of f and q. Our results generalize the above mentioned result of Işık et al. We also show for a special class of polynomials f of Carlitz rank $n \geq 1$ that if $f + x^k$ is a permutation over $\mathbb{F}_q$, with $\gcd(k+1, q-1) = 1$, then $k \geq (q-n)/(n+3)$.

1 Introduction

Let $\mathbb{F}_q$ be the finite field with $q = p^r$ elements, where $r \geq 1$ and p is a prime. Throughout we assume $q \geq 3$. We recall that $f \in \mathbb{F}_q[x]$ is a *permutation polynomial* over $\mathbb{F}_q$ if it induces a bijection from $\mathbb{F}_q$ to $\mathbb{F}_q$. If $f(x)$ and $f(x)+x$ are both permutation polynomials over $\mathbb{F}_q$, then f is called a *complete mapping*. We refer the reader to [11] for a detailed study of complete mapping polynomials over finite fields. Their use in the construction of mutually orthogonal Latin squares is described, for instance, in [9]. For various other applications, see [10, 12, 13, 14]. The paper [8] lists some recent work on complete mappings.

The Theorem 1 below was conjectured by Chowla and Zassenhaus [3] in 1968, and proven by Cohen [5] in 1990.

Theorem 1. If $d \geq 2$ and $p > (d^2 - 3d + 4)^2$, then there is no complete mapping polynomial of degree d over $\mathbb{F}_p$.

A significant generalization of this result was obtained by Cohen, Mullen and Shiue [6] in 1995, and gives a lower bound for the degree of the difference of two permutation polynomials in $\mathbb{F}_p[x]$ of the same degree d, when $p > (d^2 - 3d + 4)^2$.

Theorem 2. Suppose f and $f + g$ are monic permutation polynomials over $\mathbb{F}_p$ of degree $d \geq 3$, where $p > (d^2 - 3d + 4)^2$. If $\deg(g) = k \geq 1$, then $k \geq 3d/5$.

An alternative invariant, the so-called Carlitz rank, attached to permutation polynomials, was used by Işık, Topuzoğlu and Winterhof [8] recently to obtain a non-existence result, similar to that in Theorem 1. The concept of Carlitz rank was first introduced in [1]. We describe it here briefly. The interested reader may see [16] for details.

By a well-known result of Carlitz [2] that any permutation polynomial over

$\mathbb{F}_q$, with $q \geq 3$ is a composition of linear polynomials $ax+b$, $a, b \in \mathbb{F}_q$, $a \neq 0$, and x^{q-2}, any permutation f over $\mathbb{F}_q$ can be represented by a polynomial of the form

$$P_n(x) = \left(\dots \left((a_0 x + a_1)^{q-2} + a_2 \right)^{q-2} \dots + a_n \right)^{q-2} + a_{n+1}, \tag{1.1}$$

for some $n \geq 0$, where $a_i \neq 0$, for $i = 0, 2, \dots, n$. Note that $f(c) = P_n(c)$ holds for all $c \in \mathbb{F}_q$, however this representation is not unique, and n is not necessarily minimal. Accordingly the authors of [1] define the *Carlitz rank* of a permutation polynomial f over $\mathbb{F}_q$ to be the smallest integer $n \geq 0$ satisfying $f = P_n$ for a permutation P_n of the form (1.1), and denote it by $\mathrm{Crk}(f)$.

The representation of f as in (1.1) enables approximation of f by a fractional transformation in the following sense.

For $0 \leq k \leq n$, consider

$$R_k(x) = \frac{\alpha_{k+1} x + \beta_{k+1}}{\alpha_k x + \beta_k} \ , \tag{1.2}$$

where $\alpha_0 = 0, \alpha_1 = a_0, \beta_0 = 1, \beta_1 = a_1$, and

$$\alpha_k = a_k \alpha_{k-1} + \alpha_{k-2} \quad \text{and} \quad \beta_k = a_k \beta_{k-1} + \beta_{k-2} \tag{1.3}$$

for $k \geq 2$. The set

$$\mathcal{O}_n = \left\{ x_k : \ x_k = \frac{-\beta_k}{\alpha_k} \ , \ k = 1, \dots, n \right\} \subset \mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\} \tag{1.4}$$

is called the *set of poles* of f. The elements of $\mathcal{O}_n$ may not be distinct.

It can easily be verified that

$$f(c) = P_n(c) = R_n(c) \ \text{ for all } \ \mathrm{c} \in \mathbb{F}_q \setminus \mathcal{O}_n \ . \tag{1.5}$$

Obviously, this property is particularly useful when $\mathrm{Crk}(f)$ is small with respect to the field size. The values that f takes on $\mathcal{O}_n$ can also be expressed in terms of R_n, see [16]. In case $\alpha_n = 0$, i.e., the last *pole* $x_n = \infty$, R_n is linear. Following the terminology of [8], we define the *linearity* of $f \in \mathbb{F}_q[x]$ as $\mathcal{L}(f) = \max_{a,b \in \mathbb{F}_q} |\{c \in \mathbb{F}_q : f(c) = ac + b\}|$. Intuitively $\mathcal{L}(f)$ is large when f is a permutation polynomial of $\mathbb{F}_q$ of $\mathrm{Crk}(f) = n$, R_n is linear, and n is small with respect to q.

Now we are ready to state the main result of [8]. We remark that the Theorems 1 and 2 hold over prime fields only, while the Theorem 3 is true for any finite field.

Theorem 3. If $f(x)$ is a complete mapping over $\mathbb{F}_q$ and $\mathcal{L}(f) < \lfloor (q+5)/2 \rfloor$, then $\mathrm{Crk}(f) \geq \lfloor q/2 \rfloor$.

The purpose of this note is to obtain a lower bound for the degree of the difference between two permutation polynomials, analogous to Theorem 2, generalizing Theorem 3. In what follows we assume that f and $f+g$ are permutation polynomials over $\mathbb{F}_q$, where $g \in \mathbb{F}_q[x]$ has degree k with $1 \leq k < q-1$. We give lower bounds for k in terms of q and the Carlitz rank of f, see Theorems 2.1 and 3.1 below.

2 Degree of the difference of two permutation polynomials

Let f be a permutation polynomial over $\mathbb{F}_q$, $q \geq 3$, with $\mathrm{Crk}(f) = n \geq 1$. Suppose that f has a representation as in (1.1) and the fractional linear transformation R_n in (1.2), which is associated to f as in (1.5) is not linear, in other words α_n in (1.3) is not zero. We denote the set of all such permutations by $\mathcal{C}_{1,n}$, i.e., the set $\mathcal{C}_{1,n}$ consists of all permutation polynomials over $\mathbb{F}_q$, satisfying $\mathrm{Crk}(f) = n \geq 1$ and $\alpha_n \neq 0$. Clearly $\mathcal{L}(f) \leq n+2$, if $f \in \mathcal{C}_{1,n}$. We note that permutations $f \in \mathbb{F}_q[x]$ with $\alpha_n = 0$ behave very differently. For instance, there are examples of complete mappings over $\mathbb{F}_q$ of Carlitz rank 4 for infinitely many values of q. Indeed, the condition on the linearity of f in Theorem 3 corresponds to the case $\alpha_n = 0$. Therefore, we only consider permutations in $\mathcal{C}_{1,n}$.

We now prove our main theorem.

Theorem 2.1. *Let f and $f+g$ be permutation polynomials over $\mathbb{F}_q$, where $f \in \mathcal{C}_{1,n}$ and the degree k of $g \in \mathbb{F}_q[x]$ satisfies $1 \leq k < q-1$. Then*

$$nk + k(k-1)\sqrt{q} \geq q - \nu - n \ , \tag{2.1}$$

where $\nu = \gcd(k, q-1)$.

Proof. Since $f \in \mathcal{C}_{1,n}$, there exist $a, b, d \in \mathbb{F}_q$, such that $f(z) = R_n(z)$ for $z \in \mathbb{F}_q \setminus \mathcal{O}_n$, where

$$R_n(z) = \frac{az+b}{z+d} \ .$$

The fact that $ad - b \neq 0$ follows from (1.3).

The polynomial $f(z) + g(z)$ can be represented by $G_n(z) = R_n(z) + g(z)$ for $z \in \mathbb{F}_q \setminus \mathcal{O}_n$. Since $f+g$ is a permutation over $\mathbb{F}_q$, the map G_n is injective on $\mathbb{F}_q \setminus \mathcal{O}_n$.

For $u \in \mathbb{F}_q$ and

$$G_n(z) = \frac{az+b}{z+d} + g(z) = u \ , \tag{2.2}$$

we set

$$H_n(x) = G_n(x-d) = \frac{ax-\tilde{b}}{x} + h(x) = u \ .$$

where $\tilde{b} = ad - b \neq 0$ and $h(x) = g(x-d)$. Note that $H_n(x) = u$ for some nonzero $x \in \mathbb{F}_q$ if and only if $z \neq -d$ is a solution of Equation (2.2). Let S be the set of pairs $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ such that

$$S = \left\{ \ (x,y) \in \mathbb{F}_q^* \times \mathbb{F}_q^* \ : \ x \neq y \ \text{ and } \ H_n(x) = H_n(y) \ \right\} \ .$$

Denote the value set of H_n by V_{H_n}, i.e.,

$$V_{H_n} = \{u \in \mathbb{F}_q : \exists\, x \in \mathbb{F}_q \ \text{ with } \ H_n(x) = u\} \ .$$

Suppose that the cardinality $|S|$ of S is μ. For $u \in V_{H_n}$, we consider the inverse image; $H_n^{-1}(u) = \{x \in \mathbb{F}_q : \ H_n(x) = u\}$ and put $n_u = |H_n^{-1}(u)|$. We remark that $0 \notin H_n^{-1}(u)$ and that $x \in H_n^{-1}(u)$ if and only if x is a root of the polynomial

$$xh(x) + (a-u)x - \tilde{b} \ . \tag{2.3}$$

This shows that for any $u \in V_{H_n}$ we have $n_u \leq k+1$ as the polynomial in Equation (2.3) has degree $k+1$. We then conclude that

$$\mu = \sum_{u \in V_{H_n}} n_u(n_u - 1) \leq (k+1) \sum_{u \in V_{H_n}} (n_u - 1) \ . \tag{2.4}$$

If there exist n_u distinct elements x with $H_n(x) = u$, then there exist n_u distinct elements z with $G_n(z) = u$. Since $G_n(z)$ is injective on $\mathbb{F}_q \setminus \mathcal{O}_n$, this shows that $n_u - 1$ distinct elements z lie in the set of poles $\mathcal{O}_n$. In particular, by Equation (2.4) and the fact that $-d \in \mathcal{O}_n$ we conclude that

$$n \geq |\mathcal{O}_n| \geq 1 + \sum_{u \in V_{H_n}} (n_u - 1) \geq 1 + \frac{\mu}{k+1} \ . \tag{2.5}$$

Therefore in order to obtain a lower bound for k in terms of q and n, it is sufficient to determine μ in relation to q and k.

We can re-write the equation $H_n(x) = H_n(y)$ as

$$y(xh(x) - \tilde{b}) - x(yh(y) - \tilde{b}) = 0 \ .$$

Note that $x-y$ is a factor of $y(xh(x)-\tilde{b})-x(yh(y)-\tilde{b})$. We want to find an absolutely irreducible factor over $\mathbb{F}_q$ of the polynomial in two variables of degree $k+1$ defined by

$$\frac{y(xh(x)-\tilde{b})-x(yh(y)-\tilde{b})}{x-y} ,$$

or equivalently defined by

$$xy\frac{h(x)-h(y)}{x-y}+\tilde{b} . \tag{2.6}$$

We recall that a rational function $\ell(x)/t(x) \in \mathbb{F}_q(x)$ is called *exceptional* over $\mathbb{F}_q$ if the polynomial $\Theta_{\ell/t}$, defined by

$$\Theta_{\ell/t} = \frac{t(Y)\ell(X)-t(X)\ell(Y)}{X-Y}$$

has no absolutely irreducible factor in $\mathbb{F}_q[X,Y]$. By Theorem 5 of [4], ℓ/t is a permutation over $\mathbb{F}_q$ if it is an exceptional function over $\mathbb{F}_q$. In particular, $t(\alpha) \neq 0$ for all $\alpha \in \mathbb{F}_q$. Now we put $\ell/t = (xh(x)-\tilde{b})/x$, and conclude that the rational function in (2.6) has an absolutely irreducible factor $p(x,y)$ over $\mathbb{F}_q$. We note that $\tilde{b}$ is not zero and hence $p(x,y)$ is a factor different from $x-y$. Moreover we assume without loss of generality that $p(x,y)$ is separable; otherwise we can replace $p(x,y)$ with a separable polynomial of smaller degree.

Consider the curve $\mathcal{X}$ whose affine equation is given by $p(x,y)$ of degree $\varrho \leq k+1$. Then by [7, Theorem 9.57] the number of rational points $N(\mathcal{X})$ in $PG(2,q)$ of $\mathcal{X}$ is bounded by

$$N(\mathcal{X}) \geq q+1-(\varrho-1)(\varrho-2)\sqrt{q} \geq q+1-k(k-1)\sqrt{q} .$$

We denote by $P(X,Y,Z)$ the homogenized polynomial of $p(x,y)$, i.e.,

$$P(X,Y,Z) = Z^{\varrho}p\left(\frac{X}{Z},\frac{Y}{Z}\right) .$$

In order to find the number of affine solutions $(x:y:1)$ such that $xy \neq 0$ and $x \neq y$, we proceed as follows. From Equation (2.6) we have that $P(X,Y,Z)$ is a divisor of the homogeneous polynomial

$$XYZ^{k-1}\left(\frac{h(X/Z)-h(Y/Z)}{X-Y}\right)+\tilde{b}Z^{k+1} . \tag{2.7}$$

Hence we conclude that there is no affine solution $(x : y : 1)$ of $P(X, Y, Z)$ with $xy = 0$. We now estimate the number of rational points of $\mathcal{X}$ at infinity, i.e., the points of the form $(x : y : 0)$ for $x, y \in \mathbb{F}_q$. By Equation (2.7) the point $(x : y : 0)$ is on $\mathcal{X}$ only if

$$xy\frac{x^k - y^k}{x - y} = 0 \ .$$

This holds only if $(x : y : 0) = (0 : 1 : 0), (1 : 0 : 0)$ or $x^k = y^k$ for some $x, y \in \mathbb{F}_q^*$. Since $\nu = \gcd(k, q-1)$, the equality $x^k = y^k$ is satisfied if and only if x/y is an ν-th root of unity in $\mathbb{F}_q$. Hence there exist at most $\nu + 2$ rational points of $\mathcal{X}$ lying at infinity.

Bezout's theorem implies that there are at most $k+1$ rational points $(x : y : z)$ of $\mathcal{X}$ with $x = y$, since the degree of $\mathcal{X}$ is at most $k + 1$.

This shows that the cardinality μ of the set S satisfies

$$\mu \geq q + 1 - k(k-1)\sqrt{q} - (\nu + k + 2) \ .$$

Note that we subtract $\nu + k + 2$ instead of $\nu + k + 3$. This is because of the point $(1 : 1 : 0)$. If $(1 : 1 : 0)$ is on $\mathcal{X}$ then it is taken into account twice. If it is not on $\mathcal{X}$ then we do not have to exclude it as a point at infinity. Therefore, $\mathrm{Crk}(f) = n$ satisfies

$$\begin{aligned} n &\geq 1 + \frac{1}{k+1}(q + 1 - k(k-1)\sqrt{q} - (\nu + k + 2)) \\ &= \frac{1}{k+1}(q - k(k-1)\sqrt{q} - \nu) \ , \end{aligned}$$

by (2.5), which implies the desired result. □

For $k = 1$ (and hence $\nu = 1$) we obtain Theorem 3, i.e., the main result in [8].

Corollary 2.2. *Let $f \in \mathcal{C}_{1,n}$. If $n < (q-1)/2$, then f is not a complete mapping.*

Remark 2.3. We note that the bound given in (2.1) is non-trivial only when $q \geq k(k-1)\sqrt{q} + k + \nu + 1$.

3 The case $g(x) = cx^k$

Throughout this section we focus on the monomials $g(x) = cx^k \in \mathbb{F}_q[x]$ and $f \in \mathcal{C}_{1,n}$, where $x_n \in \mathcal{O}_n$ in (1.4) satisfies $x_n = 0$. In this particular case, the lower bound in (2.1) can be simplified significantly when $\gcd(k+1, q-1) = 1$. Let $\mathcal{C}_{2,n}$ be the set of $f \in \mathcal{C}_{1,n}$ such that the last pole x_n of f is zero.

Theorem 3.1. *Let $f(x)$ and $f(x) + cx^k$ be permutation polynomials over $\mathbb{F}_q$, where $f \in \mathcal{C}_{2,n}$, $1 \le k < q-1$, $c \in \mathbb{F}_q^*$. Put $m = \gcd(k+1, q-1)$. Then*

$$k(n+3) + (k-1)(m-1)\sqrt{q} \ge q - n \ .$$

In particular, if $m = 1$, then $k \ge (q-n)/(n+3)$.

Proof. The condition $x_n = 0$ implies that β_n in (1.3) is zero. Hence we have $R_n(x) = \frac{ax+b}{x}$ for some $a, b \in \mathbb{F}_q$, with $b \neq 0$. That is, for $x \in \mathbb{F}_q \setminus \mathcal{O}_n$ we can represent $f + cx^k$ by $G_n(x) = R_n(x) + cx^k$.

We proceed as in the proof of Theorem 2.1. The equation $G_n(x) = u$ for some $u \in \mathbb{F}_q$ becomes

$$\frac{ax+b}{x} + cx^k = u \ .$$

Then for some $x, y \in \mathbb{F}_q^*$, we have $G_n(x) = G_n(y)$ if and only if the equation

$$cx^k + \frac{b}{x} = cy^k + \frac{b}{y} \ ,$$

or equivalently the equation

$$x^k - y^k = \frac{b}{c}\left(\frac{x-y}{xy}\right) \tag{3.1}$$

holds.

We again consider the set S of pairs $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$, $x \neq y$, where (x, y) is a solution of (3.1), and denote the cardinality of S by μ. By using the argument given in the proof of Theorem 2.1, we have $n \ge 1 + \mu/(k+1)$. Hence our aim now is to express μ in terms of q and k.

Applying the change of variable $(x, y) \to (xy, y)$, Equation (3.1) becomes

$$y^k(x^k - 1) = \frac{b(x-1)}{cxy} \ .$$

Hence we are looking for the affine points $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ of the curve

$$\mathcal{X} : \ y^{k+1} = \frac{b(x-1)}{cx(x^k-1)} \ . \tag{3.2}$$

Note that in this case the solutions should not lie in the set $\{(\gamma^2, \gamma) \mid \gamma \in \mathbb{F}_q\}$. Recall that $m = \gcd(k+1, q-1)$, hence the monomial $y^{(k+1)/m}$ gives rise to a permutation over $\mathbb{F}_q^*$. Therefore, there is one-to-one correspondence between the affine solutions $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ of the curves

$$\mathcal{Y}: \; y^m = \frac{b(x-1)}{cx(x^k-1)} \;, \tag{3.3}$$

and $\mathcal{X}$ in (3.2). Equation (3.3) defines a Kummer extension. Then by using arithmetic of function fields, see [15, Proposition 3.7.3], we can estimate the number of $\mathbb{F}_q$-rational points of $\mathcal{Y}$ as follows.

For the rational function field $\mathbb{F}_q(x)$ and $\alpha \in \mathbb{F}_q$, we denote by $(x = \alpha)$ and $(x = \infty)$ the places corresponding to the zero and the pole of $x - \alpha$, respectively. Let $F = \mathbb{F}_q(x, y)$ be the function field of $\mathcal{Y}$ defined by Equation (3.3), and let $k = p^\ell t$ with $\gcd(p, t) = 1$. It is clear that the places $(x = 0)$ and $(x = \alpha)$, with $\alpha^t = 1$ and $\alpha \neq 1$, are totally ramified in F. In particular, this shows that the full constant field of F is $\mathbb{F}_q$. For the place $(x = \infty)$ we have the ramification index $e_\infty = m/\gcd(m, k) = m$, since m is a divisor of $k+1$. Moreover, for $(x = 1)$ the ramification index is given by $e_1 = m/\gcd(m, p^\ell - 1)$. Hence we conclude that the number of ramified places of $\mathbb{F}_q(x)$ in F is at most $k/p^\ell + 2$ if $\ell > 0$ and is exactly $k+1$ if $\ell = 0$. That is, the place $(x = 1)$ can be ramified only if $\ell > 0$. We consider the case $\ell = 0$, i.e. $\gcd(k, p) = 1$, where the genus of F is the largest. In this case, the ramified places are exactly

$$(x = 0), \quad (x = \infty) \quad \text{and} \quad (x = \alpha) \quad \text{with} \quad \alpha^k = 1 \text{ and } \alpha \neq 1 \;.$$

Therefore, the degree of the different divisor of $F/\mathbb{F}_q(x)$ is $(k+1)(m-1)$. Then by the Hurwitz genus formula the genus $g(F)$ of F satisfies

$$2g(F) - 2 = -2m + (k+1)(m-1) \;,$$

which implies that $g(F) = (k-1)(m-1)/2$. By the Hasse–Weil theorem the number $N(F)$ of $\mathbb{F}_q$-rational places of F is bounded by

$$N(F) \geq q + 1 - 2g(F)\sqrt{q} = q + 1 - (k-1)(m-1)\sqrt{q} \;. \tag{3.4}$$

We observe that the pole divisors $(x)_\infty$, $(y)_\infty$ of x, y are

$$(x)_\infty = mP_\infty \quad \text{and} \quad (y)_\infty = P_0 + \sum_{\alpha^k = 1, \alpha \neq 1} P_\alpha \;,$$

where P_∞, P_0, P_α are the unique places of F lying over $(x = \infty), (x = 0), (x = \alpha)$, respectively.

We remark that the curve $\mathcal{Y}$ defined by Equation (3.3) is of degree $k + m$ and has two points at infinity; namely $Q_1 = (1 : 0 : 0)$ and $Q_2 = (0 : 1 : 0)$. These are the only singular points of $\mathcal{Y}$ and Q_1 has intersection multiplicity m while Q_2 is an ordinary point of multiplicity k. Moreover, P_∞ is the unique place corresponding to Q_1, and there are k places corresponding to Q_2, which correspond to the places lying in the support of $(y)_\infty$. All the affine points in the curve $\mathcal{Y}$ defined by Equation (3.3) are non-singular and there is a one to one correspondence between these points and the places in the function field F of $\mathcal{Y}$ which do not lie in the support of pole divisors of x and y. Moreover, the fact that the zero divisors of x and y are $(x)_0 = mP_0$ and $(y)_0 = kP_\infty$, respectively, implies that the rational places not lying in the pole divisors correspond to points $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$. Therefore, Equation (3.4) implies that the number of affine points $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ of $\mathcal{Y}$ is at least $q - (k-1)(m-1)\sqrt{q} - k$.

Now we turn our attention to the curve $\mathcal{X}$ in Equation (3.2). We have seen that $\mathcal{X}$ has at least $q - (k-1)(m-1)\sqrt{q} - k$ affine points $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$. Next we estimate the number of affine points (x, y) of $\mathcal{X}$ such that (x, y) is not of the form (γ^2, γ) for some $\gamma \in \mathbb{F}_q$. By Equation (3.2), the affine point (γ^2, γ) lies on $\mathcal{X}$ if and only if γ is a root of

$$T^{k+1} \sum_{i=1}^{k} T^{2i} - \frac{b}{c} \ . \tag{3.5}$$

Since the polynomial in Equation (3.5) has degree $3k+1$, there can be at most $3k + 1$ such points. Hence the number μ of affine solutions $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ of Equation (3.2), which do not lie on the curve $x = y^2$ satisfies

$$\mu \geq q - (k-1)(m-1)\sqrt{q} - (4k+1) \ .$$

Therefore $\mathrm{Crk}(f) = n$ satisfies

$$n \geq 1 + \frac{1}{k+1}(q - (k-1)(m-1)\sqrt{q} - (4k+1)) \ .$$

□

Example 3.2. For $q = 9$, $n = 3$ and $m = 1$, the bound in Theorem 3.1 gives $k \geq 1$. Combining with Corollary 2.2 we get $k \geq 2$ as $q > 2n + 1$. Let ζ be a primitive element of $\mathbb{F}_9$ and consider the permutation polynomial $f(x) = (((x+a)^7)+b)^7+c)^7 \in \mathbb{F}_9[x]$ of Carlitz rank 3, where $a = \zeta^5$, $b = \zeta^6$ and $c = \zeta^3$. It can be checked easily that $f(x) + x^2$ is a permutation polynomial of $\mathbb{F}_9$.

Remark 3.3. As we have seen in Example 3.2, the bound in Theorem 3.1 is weaker than the one in Theorem 2.1 for $k = 1$. The reason is the change of variable $(x, y) \to (xy, y)$ in the proof of Theorem 3.1. However, a direct calculation in this specific case is possible, and gives an alternative proof for Theorem 3, which was proven in [8]. In fact, the change of variable is not needed when $k = 1$ as Equation (3.1) becomes $xy = b$. In this case, each non-zero x uniquely determines y, i.e., there exists $q-1$ distinct solutions (x, y) of $xy = b$. We also leave out the solutions (x, y) with $x = y$. We therefore obtain $\mu = q - 2$ if q is even, and $\mu = q - 3$ or $q - 1$ (depending on b being square or not) if q is odd. Then the fact that $n \geq 1 + \mu/2$ implies Corollary 2.2.

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